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Direct Estimates for Lupaş-Durrmeyer Operators

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Abstract. The generalization of the Bernstein polynomials based on Polya distribution was first considered by Stancu [14]. Very recently Gupta and Rassias [6] proposed the Durrmeyer type modification of the Lupaş operators and established some results. Now we extend the studies and here we estimate the convergence estimates, which include quantitative asymptotic formula and rate of approximation bounded variation. We also give an open problem for readers to obtain the moments using hypergeometric function.

1. Introduction

Stancu [14] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \to C[0,1]$, depending on a non-negative parameter α given by

$$P_n^{(\alpha)}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \tag{1}$$

where $p_{n,k}^{(\alpha)}(x)$ is the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x-\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, x \in [0,1].$$

In case $\alpha = 0$ these operators reduce to the classical Bernstein polynomials. For $\alpha = 1/n$ a special case of the operators (1) was considered by Lupaş and Lupaş [8], which can be represented in an alternate form as

$$P_n^{(1/n)}(f,x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) (nx)_k (n-nx)_{n-k}, \tag{2}$$

where the rising factorial is given as $(x)_n = x(x+1)(x+2)...(x+n-1)$. Recently Miclăuş [10] established some approximation results for the operators (1) and for the case (2).

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In the last few decades several new operators were constructed and their approximation properties related to convergence behaviour have been studied by several researchers. We mention here some of them due to Aral and Acar [1] who considered Bernstein-Chlodowsky-Gadjiev operators, Mahmudov and Sabancigil [9] proposed q-Bernstein Kantorovich operators, Srivastava and Gupta [12], [13] proposed a general family of integral operators and summationn-integral type operators established some convergence estimates. Very recently Gupta and Rassias [6] proposed the integral modification of the operators (2), which is based on Polya distribution as follows:

$$D_n^{(1/n)}(f,x) = \int_0^1 K_n(x,t)f(t)dt = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)\,f(t)\,dt, \tag{3}$$

where $K_n(x,t) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\,p_{n,k}(t)$ with

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}$$

and

$$p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}.$$

Some approximation properties related the present paper can be found in [5] and in the recent book by Gupta and Agarwal [4]. In this paper, we consider the operator (3) and obtain a quantitative Voronovskaja type asymptotic formula and the rate of convergence for bounded variation functions.

2. Auxiliary Results

Lemma 2.1. *(Miclăuş [10]) For $e_i = t^i, i = 0,1,2$ we have*

$$P_n^{(1/n)}(e_0,x) = 1, P_n^{(1/n)}(e_1,x) = x$$

and

$$P_n^{(1/n)}(e_2,x) = \frac{nx^2+2x-x^2}{n+1} = x^2 + \frac{2x(1-x)}{n+1}.$$

Lemma 2.2. *For $e_i = t^i, i = 0,1,2$ we have*

$$\begin{aligned}
D_n^{(1/n)}(e_0,x) &= 1, D_n^{(1/n)}(e_1,x) = \frac{nx+1}{n+2} \\
D_n^{(1/n)}(e_2,x) &= \frac{n^3x^2+5n^2x-n^2x^2+3nx+2n+2}{(n+1)(n+2)(n+3)}.
\end{aligned}$$

$$\begin{aligned}
D_n^{(1/n)}(e_3,x) &= \frac{1}{(n+2)(n+3)(n+4)}\Big(n^3x^3 + \frac{6n^4x^2(1-x)}{(n+1)(n+2)} \\
&\quad + \frac{6n^3x(1-x)}{(n+1)(n+2)} + 6n^2x^2 + \frac{12n^2x(1-x)}{n+1} + 11nx + 6\Big)
\end{aligned}$$

$$\begin{aligned}
D_n^{(1/n)}(e_4,x) &= \frac{1}{(n+2)(n+3)(n+4)(n+5)}\Big\{n^4x^4 + \frac{12n^4\left(n^2+1\right)x^3(1-x)}{(n+1)(n+2)(n+3)} \\
&\quad + \frac{12n^4(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \frac{2n^4(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)} \\
&\quad + 10n^3x^3 + \frac{60n^4x^2(1-x)}{(n+1)(n+2)} + \frac{60n^3x(1-x)}{(n+1)(n+2)} \\
&\quad + 35n^2x^2 + \frac{70n^2x(1-x)}{n+1} + 50nx + 24\Big\}
\end{aligned}$$

The proof of the above lemma easily follows by using Lemma 2.1

Remark 2.3. *In terms of hypergeometric function, we have the moments in the following complicated form: Using* $\binom{n}{k} = \frac{(-1)^k(-n)_k}{k!}$, $(a)_{n-k} = \frac{(a)_n}{(1-a-n)_k}$, $0 \le k \le n$, $\int_0^1 p_{n,k}(t)t^r dt = \frac{n!(k+r)!}{k!(n+r+1)!}$ *and* $(k+r)! = (r+1)_k.r!$ *we have*

$$\begin{aligned} D_n^{(1/n)}(e_r, x) &= (n+1)\sum_{k=0}^{n} \frac{2.n!}{(2n)!}.\frac{(-1)^k(-n)_k}{k!}(nx)_k\frac{(-1)^k(n-nx)_n}{(1-2n+nx)_k}.\frac{n!(k+r)!}{k!(n+r+1)!} \\ &= \frac{2r!.(n+1)!n!(n-nx)_n}{(n+r+1)!(2n)!}\sum_{k=0}^{n}\frac{(-n)_k(nx)_k(r+1)_k}{(1)_k(1-2n+nx)_k}.\frac{1}{k!} \\ &= \frac{2r!.(n+1)!n!(n-nx)_n}{(n+r+1)!(2n)!}\ {}_3F_2\left(-n, nx, r+1; 1, 1-2n+nx; 1\right) \\ &= \frac{r!.(n+1)!(n-nx)_n}{(n+r+1)!(n)_n}\ {}_3F_2\left(-n, nx, r+1; 1, 1-2n+nx; 1\right) \end{aligned}$$

By using the above form it may be considered as an open problem for the readers to have moments as indicated in Lemma 2.2. This problem was initially raised by Gupta in [3].

Remark 2.4. *By simple applications of Lemma 2.2, we have*

$$D_n^{(1/n)}(t-x, x) = \frac{1-2x}{n+2}$$

and

$$D_n^{(1/n)}((t-x)^2, x) = \frac{(x-x^2)(3n^2-5n-6)+2(n+1)}{(n+1)(n+2)(n+3)}.$$

$$\begin{aligned} D_n^{(1/n)}\left((t-x)^3, x\right) &= x^3\left(\frac{46n^3+54n^2-52n-48}{(n+1)(n+2)^2(n+3)(n+4)}\right) \\ &+x^2\left(\frac{-27n^3-81n^2+78n+72}{(n+1)(n+2)^2(n+3)(n+4)}\right) \\ &+x\left(\frac{23n^3+15n^2-52n-48}{(n+1)(n+2)^2(n+3)(n+4)}\right) \\ &+\frac{6}{(n+2)(n+3)(n+4)}. \end{aligned}$$

$$\begin{aligned} D_n^{(1/n)}\left((t-x)^4, x\right) &= x^4\left(\frac{27n^5-572n^4-2211n^3-1648n^2+804n+720}{(n+1)(n+2)^2(n+3)^2(n+4)(n+5)}\right) \\ &+x^3\left(\frac{1144n^4-54n^5+4422n^3+3296n^2-1608n-1440}{(n+1)(n+2)^2(n+3)^2(n+4)(n+5)}\right) \\ &+x^2\left(\frac{27n^5-754n^4-2775n^3-1634n^2+1968n+1440}{(n+1)(n+2)^2(n+3)^2(n+4)(n+5)}\right) \\ &+x\left(\frac{182n^4+564n^3-14n^2-1164n-720}{(n+1)(n+2)^2(n+3)^2(n+4)(n+5)}\right) \\ &+\frac{24}{(n+2)(n+3)(n+4)(n+5)} \end{aligned}$$

Remark 2.5. *For sufficiently large $n, C > 3$ and $x \in (0,1)$, by Remark 2.4, we have*

$$D_n^{(1/n)}\left((t-x)^2, x\right) \leq \frac{Cx(1-x)}{n},$$

and

$$\int_0^1 K_n(x,t)|t-x|dt \leq [D_n^{(1/n)}\left((t-x)^2, x\right)]^{1/2} \leq \sqrt{\frac{Cx(1-x)}{n}}.$$

Lemma 2.6. *Let $x \in (0,1)$ and $C > 3$, then for n sufficiently large, we have*

$$\lambda_n(x,y) = \int_0^y K_n(x,t)dt \leq \frac{Cx(1-x)}{n(x-y)^2}, 0 \leq y < x$$

$$1 - \lambda_n(x,z) = \int_z^1 K_n(x,t)dt \leq \frac{Cx(1-x)}{n(z-x)^2}, x < z < 1.$$

Proof of the above lemma follows easily by using Remark 2.5.

For $k \geq 1$, let us denote by $C^k = C^k[0,1]$ the subspace of $C[0,1]$ whose elements f are k-times continuosly differentiable and $f^{(k)} \in C[0,1]$ .For $f \in C^m[0,1]$, the local Taylor formula at the point $x_0 \in [0,1]$ is given by

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_m(f, x_0, x)$$

$x \in [0,1], m \geq 1$.

In [2], remainder term $R_m(f, x_0, x)$ was estimated by

$$\left|R_m(f, x_0, x)\right| \leq \frac{|x-x_0|^m}{m!}\omega\left(f^{(m)}, |x-x_0|\right).$$

Strictly related to the modulus ω is the well-known K-functional, introduced by Peetre and defined by

$$K(f,\varepsilon) = \inf\left\{\left\|f-g\right\|_\infty + \varepsilon\left\|g'\right\|_\infty : g \in C^1\right\}$$

$f \in C[0,1]$ and $\varepsilon > 0$.

The relation between modulus of continuity and corresponding K-functional is given by

$$K(f, \varepsilon/2) = \frac{1}{2}\widetilde{\omega}(f,\varepsilon)$$

for $f \in C[0,1]$,where $\widetilde{\omega}(f,\varepsilon)$ denotes the least concave majorant of $\omega(f,\varepsilon)$, see, [11].

Lemma 2.7. *[2]For $m \in \mathbb{N}$ let $f \in C^m$ and $x, x_0 \in [0,1]$. Then*

$$\left|R_m(f, x_0, x)\right| \leq \frac{|x-x_0|^m}{m!}\widetilde{\omega}\left(f^{(m)}, \frac{|x-x_0|}{m+1}\right). \tag{4}$$

3. Convergence Estimates

In this section, we present some convergence estimates of the operators $D_n^{(1/n)}(f,x)$.

Theorem 3.1. *If $f \in L^{\infty}[0,1]$ then at every point x of continuity of f we have*

$$\lim_{n\to\infty} D_n^{(1/n)}(f,x) = f(x).$$

Moreover if the function f is uniformly continuous then we have

$$\lim_{n\to\infty} \left\| D_n^{(1/n)}(f,x) - f(x) \right\|_{\infty} = 0.$$

Proof. Since $D_n^{(1/n)}(1;x) = 1$ we can write

$$D_n^{(1/n)}(f;x) - f(x) = D_n^{(1/n)}(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t)\left[f(t) - f(x)\right] dt.$$

Let $\varepsilon > 0$ be given. By the continuity of f at the point x there exists $\delta > 0$ such that $\left|f(t) - f(x)\right| < \varepsilon$ whenever $|t - x| < \delta$. For this $\delta > 0$ we can write

$$\begin{aligned} D_n^{(1/n)}(f;x) - f(x) &= (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\left(\int_{|t-x|<\delta} + \int_{|t-x|\geq\delta}\right) p_{n,k}(t)\left|[f(t) - f(x)\right] dt \\ &:= I_1 + I_2. \end{aligned}$$

It is obvious that

$$|I_1| \leq \varepsilon D_n^{(1/n)}(1,x) = \varepsilon.$$

It remains to estimate I_2. We can write

$$\begin{aligned} |I_2| &\leq 2\left\|f\right\|_{\infty}(n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_{|y-x|\geq\delta} p_{n,k}(t)\, dt \\ &\leq 2\frac{\left\|f\right\|_{\infty}}{\delta^2} D_n^{(1/n)}\left((t-x)^2, x\right). \end{aligned}$$

If we choose $\delta = \frac{1}{\sqrt[3]{n}}$ and use Remark 2.4 we have

$$|I_2| \leq 2\left\|f\right\|_{\infty}\left\{\frac{(x-x^2)(3n^2-5n-6)+2(n+1)}{n^{1/3}(n+2)(n+3)}\right\},$$

which proves the theorem. The second part of the theorem is proved similarly. □

Theorem 3.2. *Let $f'' \in C[0,1)$ and $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\left| n\left[D_n^{(1/n)}(f,x) - f(x)\right] - f'(x)(1-2x) - \frac{3f''(x)}{2}\left(x - x^2\right)\right| \\ \leq\ & \left|f'(x)\right|\frac{2}{n+2} + \frac{\left|f''(x)\right|}{8}\frac{\left(27n^2+37n+18\right)}{(n+1)(n+2)(n+3)} \\ &+\frac{1}{2}\frac{\left(x-x^2\right)\left(3n^3-5n^2-6n\right)+2\left(n^2+n\right)}{(n+1)(n+2)(n+3)}\widetilde{\omega}\left(f'', O\left(1/\sqrt{n}\right)\right) \\ =\ & o(1)\left(\left|f'(x)\right| + \left|f''(x)\right|\right) + O(1)\,\widetilde{\omega}\left(f'', O\left(1/\sqrt{n}\right)\right). \end{aligned}$$

Proof. By the local Taylor's formula there exists η lying between x and y such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + h(y,x)(y-x)^2,$$

where

$$h(y,x) := \frac{f''(\eta) - f''(x)}{2}$$

and h is a continuous function which vanishes at 0. Applying the operator $D_n^{(1/n)}$ to above equality, we obtain the equality

$$\begin{aligned} D_n^{(1/n)}(f,x) - f(x) &= f'(x)\, D_n^{(1/n)}(y-x,x) + \frac{f''(x)}{2} D_n^{(1/n)}\left((y-x)^2, x\right) \\ &\quad + D_n^{(1/n)}\left(\left(h(y,x)(y-x)^2\right), x\right) \end{aligned}$$

also we can write that

$$\begin{aligned} &\left| D_n^{(1/n)}(f,x) - f(x) - \frac{f'(x)}{n}(1-2x) - \frac{3f''(x)}{2n}\left(x-x^2\right) \right| \\ \leq\ & \left|f'(x)\right| \left| D_n^{(1/n)}(y-x,x) - \frac{(1-2x)}{n} \right| \\ & + \frac{\left|f''(x)\right|}{2} \left| D_n^{(1/n)}\left((y-x)^2, x\right) - \frac{3}{n}\left(x-x^2\right) \right| \\ & + D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2, x\right) \end{aligned}$$

Using Remark 2.4, we can write

$$\begin{aligned} &\left| n\left[D_n^{(1/n)}(f,x) - f(x)\right] - f'(x)(1-2x) - \frac{3f''(x)}{2}\left(x-x^2\right) \right| \\ \leq\ & \left|f'(x)\right| \frac{|4x-2|}{n+2} + \frac{\left|f''(x)\right|}{2} \left| \frac{\left(x-x^2\right)\left(-27n^2-37n-18\right)}{(n+1)(n+2)(n+3)} \right| \\ & + nD_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2, x\right) \end{aligned}$$

and using the fact $x \in [0,1]$ and $\max_{0\leq x\leq 1}\left(x-x^2\right) = \frac{1}{4}$ we have

$$\begin{aligned} &\left| n\left[D_n^{(1/n)}(f,x) - f(x)\right] - f'(x)(1-2x) - \frac{3f''(x)}{2}\left(x-x^2\right) \right| \\ \leq\ & \left|f'(x)\right| \frac{2}{n+2} + \frac{\left|f''(x)\right|}{8} \frac{\left(27n^2+37n+18\right)}{(n+1)(n+2)(n+3)} \\ & + nD_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2, x\right) \end{aligned}$$

To estimate the term $D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2, x\right)$ if we consider the inequality (4) for $m = 2$ then we deduce

$$\begin{aligned} D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2, x\right) &\leq \frac{1}{2} D_n^{(1/n)}\left((y-x)^2\, \widetilde{\omega}\left(f^{(2)}, \frac{\left|y-x\right|}{3}\right), x\right) \\ &\leq D_n^{(1/n)}\left((y-x)^2\, K\left(f^{(2)}, \frac{\left|y-x\right|}{6}\right), x\right) \end{aligned}$$

Let $g \in C^3$ be fixed. Then we write

$$\begin{aligned} D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2,x\right) &\leq D_n^{(1/n)}\left((y-x)^2\left[\left\|(f-g)''\right\|_\infty + \frac{|y-x|}{6}\left\|g'''\right\|_\infty\right],x\right) \\ &\leq \left\|(f-g)''\right\|_\infty D_n^{(1/n)}\left((y-x)^2,x\right) \\ &\quad + \frac{\left\|q'''\right\|_\infty}{6} D_n^{(1/n)}\left(|y-x|^3,x\right) \\ &\leq D_n^{(1/n)}\left((y-x)^2,x\right)\left(\left\|(f-g)''\right\|_\infty + \frac{\left\|g'''\right\|_\infty}{6}\frac{D_n^{(1/n)}\left(|y-x|^3,x\right)}{D_n^{(1/n)}\left((y-x)^2,x\right)}\right) \end{aligned}$$

and we have

$$\begin{aligned} D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2,x\right) &\leq D_n^{(1/n)}\left((y-x)^2,x\right)K\left(f'',\frac{1}{6}\frac{D_n^{(1/n)}\left(|y-x|^3,x\right)}{D_n^{(1/n)}\left((y-x)^2,x\right)}\right) \\ &\leq \frac{1}{2}D_n^{(1/n)}\left((y-x)^2,x\right)\widetilde{\omega}\left(f'',\frac{1}{3}\frac{\sqrt{D_n^{(1/n)}\left((y-x)^4,x\right)}}{\sqrt{D_n^{(1/n)}\left((y-x)^2,x\right)}}\right). \end{aligned}$$

Since $D_n^{(1/n)}\left((y-x)^4,x\right) = O\left(1/n^2\right)$ and $D_n^{(1/n)}\left((y-x)^2,x\right) = O(1/n)$ we get

$$D_n^{(1/n)}\left(\left|h(y,x)\right|(y-x)^2,x\right) \leq \frac{1}{2}\frac{\left(x-x^2\right)\left(3n^3-5n^2-6n\right)+2\left(n^2+n\right)}{(n+1)(n+2)(n+3)}\widetilde{\omega}\left(f'',O\left(1/\sqrt{n}\right)\right),$$

which completes the proof. □

We denote by $B_D(0,1)$ the class of absolutely continuous functions f on $(0,1)$ having a derivative f' on $(0,1)$, which concide a.e. with a function which is of bounded variation on every subinterval of $(0,1)$. The functions $f \in B_D(0,1)$ possess the representation

$$f(x) = f(c) + \int_c^x \phi(t)dt, x \geq c > 0.$$

Theorem 3.3. *Let $f \in B_D(0,1)$ and $x \in (0,1)$, then for n sufficiently large and $C > 3$, we have*

$$\begin{aligned} \left|D_n^{(1/n)}(f,x) - f(x)\right| &\leq \sqrt{\frac{Cx(1-x)}{n}}\frac{[f'(x+)-f'(x-)]}{2} + \frac{(1-2x)}{n+2}\frac{[f'(x+)+f'(x-)]}{2} \\ &\quad + \frac{C}{nx(1-x)}\left(\sum_{k=1}^{[\sqrt{n}]} V_{x-x/k}^{x+(1-x)/k}((f')_x) + \frac{1}{\sqrt{n}}V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}((f')_x)\right), \end{aligned}$$

where $V_a^b f$ is the total variation of f on $[a,b]$ and the auxiliary function is given by

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x; \\ 0, & t = x; \\ f(t) - f(x+), & x < t < 1. \end{cases}$$

Proof. By mean value theorem and using the methods as given in [4], we have

$$\begin{aligned}\left|D_n^{(1/n)}(f,x)-f(x)\right| &\le \frac{|f'(x+)+f'(x-)|}{2}\int_0^1\left(\int_x^t K_n(x,t)du\right)dt\\ &+\frac{|f'(x+)-f'(x-)|}{2}\int_0^1\left(\int_x^t K_n(x,t)\,(sign(u-x)du)\right)dt\\ &+\left|\int_0^1\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right|\\ &\le \frac{|f'(x+)+f'(x-)|}{2}D_n^{(1/n)}(t-x,x)\\ &+\frac{|f'(x+)-f'(x-)|}{2}[D_n^{(1/n)}((t-x)^2,x)]^{1/2}\\ &+\left|\left(\int_x^1+\int_0^x\right)\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right|\end{aligned}$$

Using Remark 2.4 and Remark 2.5, we have

$$\begin{aligned}\left|D_n^{(1/n)}(f,x)-f(x)\right| &\le \frac{|f'(x+)+f'(x-)|}{2}\frac{(1-2x)}{n+2}+\frac{|f'(x+)-f'(x-)|}{2}\sqrt{\frac{Cx(1-x)}{n}}\\ &+\left|\int_x^1\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right|\\ &+\left|\int_0^x\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right| \qquad (5)\end{aligned}$$

As $\int_a^b d_t(\lambda_n(x,t)\le 1$ for each $[a,b]\subseteq[0,1]$, using Lemma 2.6 with $y=x-x/\sqrt{n}$, we get

$$\begin{aligned}&\left|\int_0^x\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right|=\left|\int_0^x\left(\int_x^t (f')_x(u)du\right)d_t(\lambda_n(x,t))\right|\\ \le\ &\left(\int_0^y+\int_y^x\right)|(f')_x(t)|.|\lambda_n(x,t)|dt\\ \le\ &\frac{Cx(1-x)}{n}\int_0^y V_t^x((f')_x)\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}V^x_{x-x/\sqrt{n}}((f')x)\end{aligned}$$

Setting $u=x/(x-t)$, we have

$$\int_0^y V_t^x((f')_x)\frac{1}{(x-t)^2}dt=\frac{1}{x^2}\int_1^{\sqrt{n}}V^x_{x-x/u}((f')_x)du\le\frac{1}{x^2}\sum_{k=1}^{[\sqrt{n}]}V^x_{x-x/k}((f')_x).$$

Thus

$$\begin{aligned}&\left|\int_0^x\left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right|\\ \le\ &\frac{Cx(1-x)}{nx}\sum_{k=1}^{[\sqrt{n}]}V^x_{x-x/k}((f')_x)+\frac{x}{\sqrt{n}}V^x_{x-x/\sqrt{n}}((f')x) \qquad (6)\end{aligned}$$

Finally

$$\begin{aligned} & \left|\int_x^1 \left(\int_x^t K_n(x,t)(f')_x(u)du\right)dt\right| \\ = & \left|\int_z^1 \left(\int_x^t (f')_x(u)du\right)K_n(x,t)dt + \int_x^z \left(\int_x^t (f')_x(u)du\right)d_t(1-\lambda_n(x,t))dt\right| \\ \leq & \frac{C}{n(1-x)}\int_1^{\sqrt{n}} V_x^{x+(1-x)/\sqrt{n}}((f')_x)du + \frac{(1-x)}{\sqrt{n}} V_x^{+(1-x)/\sqrt{n}}((f')_x) \\ \leq & \frac{C}{n(1-x)}\sum_{k=1}^{[\sqrt{n}]} V_x^{x+(1-x)/\sqrt{n}}((f')_x)du + \frac{(1-x)}{\sqrt{n}} V_x^{+(1-x)/\sqrt{n}}((f')_x) \qquad (7) \end{aligned}$$

Combining the estimates (5), (6) and (7), and using $(1-x)^2 + x^2 \leq 1$, we are let to the conclusion of the theorem. □

Remark 3.4. *Very recently Gupta and Tachev [7] considered the combinations of certain Durrmeyer type operators and Verma et al. [15] considered the Stancu variant of some Durrmeyer type operators. On can consider the methods here to apply combinations and Stancu variants of these operators. At this moment it is not possible to establish analogous results. We may consider such results in future.*

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